VARIOUS CALCULATIONS FOR THE CONTINUOUS ELECTRICAL CONDUCTIVITY PROBLEM

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This paper is the joint work of James Hart and Matt Robinson. It contains all of the calculations of the various ideas with which we were working. It involves a brief attempt at finding a Green's function for an assumed radial conductivity. The latter two sections are trying to find Green's functions or representations of solutions (or weak solutions) to the conductivity equation by iteratively defining solutions to Poisson's equation and applying a limiting procedure.

1. Radial Conductivity

For the electrical conductance problem in $\mathbb{R}^n$, we have $D(\gamma Du) = 0$. Where both $\gamma, u : \mathbb{R}^n \to \mathbb{R}$. Now, for the following calculation we assume that γ, u are both sufficiently smooth, and impose that γ, u are radial from the origin.

Then,

$$r = |x| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$$

and

$$\frac{\partial r}{\partial x_j} = \frac{x_j}{r}.$$

Now define, $\mu(r) := \gamma(x)$ and $v(r) := u(x)$. So we obtain,

$$Du(x) = v'(r)\frac{x}{r}.$$

After substituting, our PDE turns into,

$$\begin{aligned} D(\gamma Du) &= D \cdot (\mu(r)v'(r)\frac{x}{r}) \\ &= \mu'(r)v'(r)\frac{x \cdot x}{r^2} + \mu(r)v''(r)\frac{x \cdot x}{r^2} + \mu(r)v'(r)\frac{n}{r} - \mu(r)v'(r)\frac{x \cdot x}{r^3} \\ &= \mu'(r)v'(r) + \mu(r)v''(r) + \mu(r)v'(r)\frac{n-1}{r} = 0. \end{aligned}$$

Multiplying by r^{n-1},

$$r^{n-1}\mu'(r)v'(r) + r^{n-1}\mu(r)v''(r) + r^{n-2}(n-1)\mu(r)v'(r) = 0$$

And thus for $n > 1$

$$\frac{d}{dr}(r^{n-1}\mu(r)v'(r)) = 0,$$

yielding

$$r^{n-1}\mu(r)v'(r) = \text{constant}.$$

So we have that

$$v(r) = k\int_a^r \frac{1}{s^{n-1}\mu(s)}ds,$$

for some constants k and a. So define the "fundamental solution" to be

$$\text{“}H(x)\text{”} = H(|x|) = \frac{1}{n\omega(n)}\int_a^{|x|}\frac{1}{s^{n-1}\mu(s)}ds,$$

where $\omega(n)$ is the volume of the unit ball in $\mathbb{R}^n$. Similarly for some fixed $y \in \mathbb{R}^n$, $x \mapsto H(x-y) = H(|x-y|)$ is also γ-harmonic.

Notice, that if we take $\gamma \equiv 1$ we obtain the Newtonian potential (up to an additive constant).

Now suppose $\Omega \subset \mathbb{R}^n$ is open, bounded and that $\partial\Omega$ is C^1. Suppose $u \in C^2(\overline{\Omega})$, and H is defined as above; that is, $D(\gamma DH) = 0$, if γ is assumed to be radial). Fix $x \in \Omega$ and let $B(x,\epsilon) \subset \Omega$ for some sufficiently small $\epsilon > 0$ (since $H(y-x)$ has singularity when $y = x$). Define $V_\epsilon := \Omega \setminus B(x,\epsilon)$, so $\partial V_\epsilon = \partial B(x,\epsilon) \cup \partial\Omega$. So the use of Green's theorem gives us the following:

$$\begin{aligned}\int_{V_\epsilon} u(y)D(\gamma(y-x)DH(y-x)) - H(y-x)D(\gamma(y-x)Du(y)dy = \\ -\int_{V_\epsilon} Du(y)\cdot\gamma(y-x)DH(y-x) - DH(y-x)\cdot\gamma(y-x)Du(y)dy \\ +\int_{\partial V_\epsilon} u(y)\gamma(y-x)\frac{\partial H}{\partial\nu}(y-x) - H(y-x)\gamma(y-x)\frac{\partial u}{\partial\nu}(y)d\sigma(y),\end{aligned}$$

where ν is the outward unit normal. Now consider,

$$\begin{aligned}&|\int_{\partial B(x,\epsilon)} H(y-x)\gamma(y-x)\frac{\partial u}{\partial\nu}(y)d\sigma(y)| \\ &= |\int_{\partial B(x,\epsilon)} H(\epsilon)\gamma(\epsilon)\frac{\partial u}{\partial\nu}(y)d\sigma(y)| \\ &\le H(\epsilon)\gamma(\epsilon)\int_{\partial B(x,\epsilon)}|Du(y)\cdot\nu|d\sigma(y) \\ &= H(\epsilon)\gamma(\epsilon)n\omega(n)\epsilon^{n-1}\sup_{\partial B(x,\epsilon)}|Du|\end{aligned}$$

Now,

$$H(\epsilon) = \frac{1}{n\omega(n)}\int_a^\epsilon \frac{1}{s^{n-1}\gamma(s)}ds,$$

so

$$\epsilon^{n-1}n\omega(n)H(\epsilon) = \epsilon^{n-1}\int_a^\epsilon\frac{1}{s^{n-1}\gamma(s)}ds \le \epsilon^{n-1}M\int_a^\epsilon\frac{1}{s^{n-1}}ds = M\epsilon^{n-1}(\frac{1}{\epsilon^{n-2}}-\frac{1}{a^{n-2}}) \to 0 \text{ as } \epsilon \to 0.$$

Hence,

$$|\int_{\partial B(x,\epsilon)} H(y-x)\gamma(y-x)\frac{\partial u}{\partial \nu}(y)d\sigma(y)| \to 0 \text{ as } \epsilon \to 0.$$

Next, we'll consider

$$\int_{\partial B(x,\epsilon)} u(y)\gamma(y-x)\frac{\partial H}{\partial \nu}(y-x)d\sigma(y)$$

First we'll note that

$$DH(y-x) = \frac{1}{n\omega(n)}\frac{y-x}{|y-x|^n\gamma(y-x)}$$

and ν, the outward pointing unit normal is

$$\nu = -\frac{y-x}{\epsilon} \qquad (\epsilon = |y-x|).$$

Thus,

$$\frac{\partial H}{\partial \nu}(y-x) = DH(y-x)\cdot\nu = \frac{1}{n\omega(n)}\frac{y-x}{\epsilon^n\gamma(\epsilon)}\cdot\frac{x-y}{\epsilon} = -\frac{1}{n\omega(n)}\frac{\epsilon^2}{\epsilon^{n+1}\gamma(\epsilon)} = -\frac{1}{n\omega(n)}\frac{1}{\epsilon^{n-1}\gamma(\epsilon)}.$$

Thus,

$$\begin{aligned}\int_{\partial B(x,\epsilon)} u(y)\gamma(y-x)\frac{\partial H}{\partial \nu}(y-x)d\sigma(y) &= \frac{1}{n\omega(n)}\int_{\partial B(x,\epsilon)} u(y)\gamma(\epsilon)\frac{-1}{\epsilon^{n-1}\gamma(\epsilon)}d\sigma(y)\\ &= -\frac{1}{n\omega(n)\epsilon^{n-1}}\int_{\partial B(x,\epsilon)} u(y)d\sigma(y) \to -u(x) \text{ as } \epsilon \to 0.\end{aligned}$$

Now, by recalling that $D(\gamma DH) = 0$ away from x our above integral equation turns into

$$\begin{aligned}-\int_{V_\epsilon} H(y-x)D(\gamma(y-x)Du(y))dy = &\int_{\partial\Omega} u(y)\gamma(y-x)\frac{\partial H}{\partial \nu}(y-x)d\sigma(y)\\ &-\int_{\partial B(x,\epsilon)} u(y)\gamma(y-x)\frac{\partial H}{\partial \nu}(y-x)d\sigma(y)\\ &-\int_{\partial\Omega} H(y-x)\gamma(y-x)\frac{\partial u}{\partial \nu}(y)d\sigma(y)\\ &+\int_{\partial B(x,\epsilon)} H(y-x)\gamma(y-x)\frac{\partial u}{\partial \nu}(y)d\sigma(y);\end{aligned}$$

so taking $\epsilon \to 0$ and applying our two previously calculated integral equalities, we obtain

$$-\int_\Omega H(y-x)D(\gamma(y-x)Du(y))dy = \int_{\partial\Omega} u(y)\gamma(y-x)\frac{\partial H}{\partial \nu}(y-x) - H(y-x)\gamma(y-x)\frac{\partial u}{\partial \nu}(y)d\sigma(y) + u(x)$$

or rearranged

$$u(x) = -\int_\Omega H(y-x)D(\gamma(y-x)Du(y))dy + \int_{\partial\Omega}\gamma(y-x)[H(y-x)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial H}{\partial \nu}(y-x)]d\sigma(y)$$

Let's assume Ω as before and $u \in C^2(\overline{\Omega})$. Define a corrector function $\phi^x(y)$ such that it satisfies

$$\begin{cases} D(\gamma D\phi^x) = 0 & \text{in } \Omega \\ \phi^x = H(y-x) & \text{on } \partial\Omega \end{cases}.$$

Applying Green's theorem to ϕ^x, u in a similar fashion,

$$\int_\Omega u(y)D(\gamma(y-x)D\phi^x(y)) - \phi^x(y)D(\gamma(y-x)Du(y))dy = \int_{\partial\Omega} \gamma(y-x)[u(y)\frac{\partial \phi^x}{\partial \nu}(y - \phi^x(y)\frac{\partial u(y)}{\partial \nu}]d\sigma(y)),$$

and now after inserting the requirements of ϕ^x,

$$-\int_\Omega \phi^x(y)D(\gamma(y-x)Du(y))dy = \int_{\partial\Omega} \gamma(y-x)[u(y)\frac{\partial \phi^x}{\partial \nu}(y) - H(y-x)\frac{\partial u}{\partial \nu}(y)]d\sigma(y)$$

Adding to our previous integral result for $u(x)$ we get,

$$\begin{aligned} u(x) = &-\int_\Omega H(y-x)D(\gamma(y-x)Du(y))dy + \int_\Omega \phi^x(y)D(\gamma(y-x)Du(y))dy \\ &+ \int_{\partial\Omega} \gamma(y-x)[H(y-x)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial H}{\partial \nu}(y-x)]d\sigma(y) \\ &+ \int_{\partial\Omega} \gamma(y-x)[u(y)\frac{\partial \phi^x}{\partial \nu}(y) - H(y-x)\frac{\partial u}{\partial \nu}(y)]d\sigma(y) \end{aligned}$$

That is,

$$\begin{aligned} u(x) = &-\int_\Omega [H(y-x) - \phi^x(y)]D(\gamma(y-x)Du(y))dy \\ &- \int_{\partial\Omega} \gamma(y-x)u(y)[\frac{\partial H}{\partial \nu}(y-x) - \frac{\partial \phi^x}{\partial \nu}(y)]d\sigma(y) \end{aligned}$$

Now, let's define our "Green's function", $G(x,y)$, to be

$$G(x,y) := H(y-x) - \phi^x(y)$$

Thus,

$$u(x) = -\int_\Omega G(x,y)D(\gamma(y-x)Du(y))dy - \int_{\partial\Omega} \gamma(y-x)u(y)\frac{\partial G}{\partial \nu}(x,y)d\sigma(y)$$

Now if we suppose that u solves the Dirichlet problem,

$$\begin{cases} D(\gamma Du) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

we obtain that

$$u(x) = -\int_{\partial\Omega} \gamma(y-x)g(y)\frac{\partial G}{\partial \nu}(x,y)d\sigma(y).$$

[See [1] for an analogous derivation for the Laplacian.]

Now, given a nice enough domain we can calculate $G(x,y)$ in a similar fashion as the Green's function associated with the Laplacian. This calculation for $u(x)$ is ONLY valid at the point x, which is the center for which γ varies radially outward. Since, we have this value at the center, it seems reasonable that one might be able to approximate u in a small ball about x, though we have not explored much into this, nor do we personally know of any techniques that might be useful in such an exploration.

2. Calculus with Iteration

For the following, let $\Omega \subset \mathbb{R}^n$ be open, bounded and have smooth boundary and let g be a sufficiently smooth boundary condition. Now we define a sequence of functions in $C^2(\Omega)$, with v_0 satisfying

$$\begin{cases} -\Delta v_0 = 0 & \text{in } \Omega \\ v_0 = g & \text{on } \partial\Omega \end{cases}$$

and then iteratively, we define $v_n, n \geq 1$ by

$$\begin{cases} -\Delta v_n = D\varphi \cdot Dv_{n-1} & \text{in } \Omega \\ v_n = g & \text{on } \partial\Omega \end{cases}.$$

Now, if we let $\varphi = \log(\gamma)$, and take the limit as $n \to \infty$, we obtain the electrical conductivity equation under Dirichlet boundary conditions. The above two PDE's are just Laplace's equations and Poisson's equation with Dirichlet conditions and so we have a well defined Green's function and representation formula. For v_0 we have the following:

$$v_0(x) = -\int_{\partial\Omega} g(y)\frac{\partial G}{\partial \nu}(x,y)d\sigma(y),$$

where $G(x,y)$ is our Green's function for the Laplacian and ν is our outward pointing normal. Similarly for v_n we have:

$$v_n = \int_{\Omega} G(x,y)D\varphi \cdot Dv_{n-1}dy - \int_{\partial\Omega} g(y)\frac{\partial G}{\partial \nu}(x,y)d\sigma(y).$$

We would like to take the limit of both sides, to give us a representation formula for the limiting function v which would solve the electrical conductivity equation. In order to bring that limit on the inside we need to analyze the integrand, so let's look at Dv_n. Now,

$$\begin{aligned}
Dv_n(x) &= D_x \int_\Omega G(x,y) D\varphi \cdot Dv_{n-1} dy - D_x \int_{\partial\Omega} g(y) \frac{\partial G}{\partial \nu}(x,y) d\sigma(y) \\
&= \int_\Omega D_x G(x,y) D\varphi \cdot Dv_{n-1} dy - \int_{\partial\Omega} g(y) D_x \cdot D_y G(x,y) \nu(y)(x,y) d\sigma(y) \\
&= \int_\Omega D_y G(x,y) D\varphi \cdot Dv_{n-1} dy - \int_{\partial\Omega} g(y) D_y \cdot D_y G(x,y) \nu(y)(x,y) d\sigma(y) \\
&= \int_\Omega DG(x,y) D\varphi \cdot Dv_{n-1} dy - \int_{\partial\Omega} g(y) \Delta G(x,y) \nu(y) d\sigma(y) \\
&= -\int_\Omega \Delta G(x,y) \varphi Dv_{n-1} + DG(x,y) \varphi \Delta v_{n-1} dy \\
&\quad + \int_{\partial\Omega} \varphi Dv_{n-1} \frac{\partial G}{\partial \nu}(x,y) d\sigma(y) - \int_{\partial\Omega} g(y) \Delta G(x,y) d\sigma(y) \\
&= \varphi(x) Dv_{n-1}(x) + \int_\Omega DG(x,y) \varphi D\varphi \cdot Dv_{n-2} dy \\
&\quad + \int_{\partial\Omega} \varphi Dv_{n-1} \frac{\partial G}{\partial \nu}(x,y) d\sigma(y) - \int_{\partial\Omega} g(y) \Delta G(x, y_\nu(y) d\sigma(y) \\
&\vdots \\
&= \sum_{j=1}^{n} \frac{\varphi^j(x)}{j!} Dv_{n-j}(x) + \int_{\partial\Omega} [\sum_{j=1}^{n} \frac{\varphi^j(y)}{j!} Dv_{n-j}(y)] \frac{\partial G}{\partial \nu}(x,y) d\sigma(y) \\
&\qquad - \int_{\partial\Omega} g(y) \Delta G(x,y) \nu(y) d\sigma(y).
\end{aligned}$$

Notice that since Dv_n is dependent on $Dv_k, k \in \{0, 1, \ldots, n-1\}$, we could then apply the same formula to Dv_{n-1} and it would be dependent on $Dv_k, k \in \{0, 1, \ldots, n-2\}$. Hence, by repeating this, we see that Dv_n is only dependent on Dv_0. Actually finding this form of the representation has been quite difficult. Dv_n gets ugly really quickly if it's written out explicitly. Here are the first 3 calculated Dv_n, with $B := \int_{\partial\Omega} g(y) \Delta G(x,y) \nu(y) d\sigma(y)$:

$$Dv_1 = \frac{\phi}{1!} Dv_0 + \int_{\partial\Omega} \frac{\phi}{1!} Dv_0 \frac{\partial G}{\partial \nu} d\sigma - B$$

$$\begin{aligned}
Dv_2 = \phi^2 Dv_0 (\frac{1}{1!1!} + \frac{1}{2!}) - (\frac{\phi}{1!} + 1) B + (\frac{1}{1!1!} + \frac{1}{2!}) \int_{\partial\Omega} \phi^2 Dv_0 \frac{\partial G}{\partial \nu} d\sigma \\
+ \frac{\phi}{1!} \int_{\partial\Omega} \frac{\phi}{1!} Dv_0 \frac{\partial G}{\partial \nu} d\sigma - \int_{\partial\Omega} \frac{\phi}{1!} B \frac{\partial G}{\partial \nu} d\sigma + \int_{\partial\Omega} \frac{\phi}{1!} \frac{\partial G}{\partial \nu} \int_{\partial\Omega} \frac{\phi}{1!} Dv_0 \frac{\partial G}{\partial \nu} d\sigma d\sigma
\end{aligned}$$

$$
\begin{aligned}
Dv_3 = &\ \phi^3 Dv_0[\frac{1}{1!1!1!}+\frac{1}{1!2!}+\frac{1}{2!1!}+\frac{1}{3!}] - [\phi^2(\frac{1}{1!1!}+\frac{1}{2!})+\frac{\phi}{1!}+1]B \\
&+\phi^2(\frac{1}{1!1!}+\frac{1}{2!})\int \frac{\phi}{1!}Dv_0\frac{\partial G}{\partial \nu}d\sigma + \frac{\phi}{1!}(\frac{1}{1!1!}+\frac{1}{2!})\int \phi^2 Dv_0 \frac{\partial G}{\partial \nu}d\sigma \\
&+\int \phi^3[\frac{1}{1!1!1!}+\frac{1}{1!2!}+\frac{1}{2!1!}+\frac{1}{3!}]Dv_0\frac{\partial G}{\partial \nu}d\sigma \\
&+\frac{\phi}{1!}\int \frac{\phi}{1!}\frac{\partial G}{\partial \nu}\int \frac{\phi}{1!}Dv_0\frac{\partial G}{\partial \nu}d\sigma d\sigma + \int \phi^2(\frac{1}{1!1!}+\frac{1}{2!})\frac{\partial G}{\partial \nu}\int \frac{\phi}{1!}Dv_0\frac{\partial G}{\partial \nu}d\sigma d\sigma \\
&+\int \frac{\phi}{1!}(\frac{1}{1!1!}+\frac{1}{2!})\frac{\partial G}{\partial \nu}\int \phi^2 Dv_0\frac{\partial G}{\partial \nu}d\sigma d\sigma + \int \frac{\phi}{1!}\frac{\partial G}{\partial \nu}\int \frac{\phi}{1!}\frac{\partial G}{\partial \nu}\int \frac{\phi}{1!}Dv_0\frac{\partial G}{\partial \nu}d\sigma d\sigma d\sigma \\
&-\frac{\phi}{1!}\int \frac{\phi}{1!}B\frac{\partial G}{\partial \nu}d\sigma - \int (\phi^2(\frac{1}{1!1!}+\frac{1}{2!})+\frac{\phi}{1!})B\frac{\partial G}{\partial \nu}d\sigma - \int \frac{\phi}{1!}\frac{\partial G}{\partial \nu}\int \frac{\phi}{1!}B\frac{\partial G}{\partial \nu}d\sigma d\sigma
\end{aligned}
$$

and Dv_4 would have its own page had it been included. We are unsure as of yet, if we can find a somewhat nice form for Dv_n and if that form converges.

3. Weak Solution Iteration

In this section, we are using the same iteration technique as in the previous, except we will be working in a weak sense. Because of this, we won't have a Green's function or a nice representation formula, but we can use a lot of the same calculus techniques that we used previously, because of the test functions being paired with the our set of PDE's. Let $\Omega \subset \mathbb{R}^n$ be open, bounded and have smooth boundary. Then we recursively define a sequence, u_n, to be weak solutions to Poisson's and Laplace's equations under Dirichlet boundary conditions. That is, define u_0 such that

$$0 = \int_\Omega -\Delta u_0 v dx = \int_\Omega Du_0 \cdot Dv dx \text{ for all } v \in H_0^1(\Omega)$$

and $u_0 = g$ on $\partial\Omega$ in a trace sense. Then we define u_n by

$$\int_\Omega -\Delta u_n v dx = \int_\Omega D\varphi \cdot Du_{n-1} v dx \text{ for all } v \in H_0^1(\Omega)$$

and $u_n = g$ on $\partial\Omega$ also in a trace sense. Now notice that by taking the limit at $n \to \infty$ and assuming (for now) commutativity of the limit and integral, we would obtain a limiting function, u such that

$$\int_\Omega -\Delta u v dx = \int_\Omega D\varphi \cdot Du v dx \text{ for all } v \in H_0^1(\Omega)$$

and u would also equal g in trace on the boundary. Hence u would be a weak solution to the conductivity equation; that is, if $\varphi = \log(\gamma)$ and

$$\begin{cases} -\Delta u = D\varphi \cdot Du & \text{in } \Omega \\ \quad u = g & \text{on } \partial\Omega \end{cases}$$

in a weak sense.

We would like to find a representation of u (or at least Du) in terms of the weak solutions that we do know, more specifically in terms of u_0, the weak harmonic function. Now, we consider

$$\begin{aligned}
\int_\Omega Du_n \cdot Dv dx &= \int_\Omega D\varphi \cdot Du_{n-1} v dx \\
&= -\int_\Omega \varphi(\Delta u_{n-1} v + Du_{n-1} \cdot Dv) dx \\
&= \int_\Omega -\Delta u_{n-1}(\varphi v) dx - \int_\Omega \varphi Du_{n-1} \cdot Dv dx \\
&= \int_\Omega D\varphi \cdot Du_{n-2}(\varphi v) dx - \int_\Omega \varphi Du_{n-1} \cdot Dv dx \\
&= -\int_\Omega \varphi(\Delta u_{n-2}(\varphi v) + Du_{n-2} \cdot D\varphi v + \varphi Du_{n-2} \cdot Dv) dx - \int_\Omega \varphi Du_{n-1} \cdot Dv dx \\
&= -\frac{1}{2} \int_\Omega \Delta u_{n-2}(\varphi^2 v) dx - \int_\Omega [\frac{\varphi^2}{2} Du_{n-2} + \frac{\varphi}{1} Du_{n-1}] \cdot Dv dx \\
&\vdots \\
&= \frac{1}{n!} \int_\Omega -\Delta u_0(\varphi^n v) dx - \int_\Omega \sum_{j=1}^n \frac{\varphi^j}{j!} Du_{n-j} \cdot Dv dx \\
&= -\sum_{j=1}^n \int_\Omega \frac{\varphi^j}{j!} Du_{n-j} \cdot Dv dx
\end{aligned}$$

for all $v \in H_0^1(\Omega)$. Note that we used the fact that $\varphi^j v$ is at least still in $H_0^1(\Omega)$ and so we could apply our weak definitions. Therefore we end up with the following equation:

$$\sum_{j=0}^n \int_\Omega \frac{\varphi^j}{j!} Du_{n-j} \cdot Dv dx = 0,$$

or that

$$Du_n + \varphi Du_{n-1} + \frac{\varphi^2}{2} Du_{n-2} + \cdots + \frac{\varphi^n}{n!} Du_0 = 0$$

in a distributional sense. We feel like this sequence $\{Du_n\}$ should converge in some sense, but we have not been able to give a proof of such a result. Similarly to section two, it seems like we can write Du_n in terms of just Du_0, but there are more technicalities to be considered in this case.

References

[1] Lawrence C. Evans, *Partial Differential Equations.* American Mathematical Society, 4th Edition, 1998.